On a 3D Model with Anisotropic, Rotating Convection and Phase Changes for DA

Onno Bokhove, Steve Tobias, Tom Kent
School of Mathematics, University of Leeds, UK

1 Introduction

- High-resolution (convective-scale) NWP models are becoming the norm: more dynamical processes such as convection, cloud formation, & small-scale gravity waves, are resolved explicitly.
- DA techniques need to evolve in order to keep up with the developments in high-resolution NWP.
- It may be unfeasible (and even undesirable) to investigate the potential of DA schemes on state-of-the-art NWP models. Idealised models have been employed that:
  - capture some fundamental features of dynamics,
  - are computationally inexpensive to implement, and
  - allow an extensive investigation of the proposed scheme.
- A hierarchy of "toy" models, e.g., Lorenz' model (L2005), QG/BV model have been employed in DA, including:
  - a "convective-scale" 1.5D shallow-layer model (Kent et al. 2015, this DA workshop).
- Here, we propose to add a 3D rotating convection model to this hierarchy.

2 Boussinesq Parent Model

- Following Julien et al. (2006), rotating Boussinesq equations are scaled with $\delta p^*, \delta\rho^*$ buoyancy $B = g|\delta\rho^*|/\rho_r^*$, horizontal and vertical length scales L and L_z (ratio $A_z = L_z/L$), and velocity scales U and $L_z U/L$.
- The resulting dimensionless model reads:

$$
\begin{aligned}
D_t \mathbf{u}_H + \frac{1}{Ro}\hat{\mathbf{z}} \times \mathbf{u} &= -P\nabla_H p + \frac{1}{Re}\nabla^2 \mathbf{u}_H && \text{(1a)}\\
A_z D_t w &= -\frac{P}{A_z}\partial_z p + \Gamma b + \frac{1}{Re}\nabla^2 w && \text{(1b)}\\
D_t\Big(b - \frac{1}{\Gamma F r^2}\bar\rho\Big) &= \frac{1}{Pe}\nabla^2 b && \text{(1c)}\\
\nabla\cdot\mathbf{u} &= 0 && \text{(1d)}
\end{aligned}
$$

- with velocity $\mathbf{u} = (\mathbf{u}_H, w)$, buoyancy $b = -g\rho/\rho_r$, background density $\bar\rho(z)$,
- Rossby $Ro = U/(2\Omega L)$, Froude $Fr = U/(N_0 L)$ (buoyancy frequency N_0), Euler $P = \delta p^*/(\rho_r^* U^2)$, buoyancy $\Gamma = BL/U^2$, and Reynolds Re and Peclet Pe numbers.

3 3D Model of Rotationally Constrained Convection

- Using a multi-scale, singular expansion in $Ro = \epsilon$ with $P = 1/\epsilon^2, \Gamma = \hat\Gamma/\epsilon, A_z = O(1), \Gamma F r^2 = O(1), b \to b_0 + \epsilon b_1, p \to \epsilon p, \partial_z \to \epsilon\partial_z$ a non-hydrostatic rotationally constrained model Julien et al. (2006) derive is:

$$
\begin{aligned}
\partial_t \zeta &= -J(\psi,\zeta) + \partial_z w_0 + \underline{\frac{1}{Re}\nabla_H^2\zeta} && \text{(2a)}\\
\partial_t w_0 &= -J(\psi, w_0) \underline{\underline{-\partial_z\psi + \hat\Gamma b_1}} + \underline{\frac{1}{Re}\nabla_H^2 w_0} && \text{(2b)}\\
\partial_t b_1 &= -J(\psi, b_1) - w_0\partial_z(\bar b_0 - \bar\rho/\hat\Gamma) + \underline{\frac{1}{Pe}\nabla_H^2 b_1} && \text{(2c)}\\
\zeta &= \nabla_H^2\psi && \text{(2d)}
\end{aligned}
$$

- with horizontal Laplacian $\nabla_H^2 = \partial_x^2 + \partial_y^2$, (leading order) vertical velocity w_0,
- slowly evolving or constant buoyancy $\bar b_0$, next order buoyancy b_1, $p = \psi$, Jacobian $J(\psi,\zeta) \equiv \partial_x\psi\partial_y\zeta - \partial_x\zeta\partial_y\psi$ etc., and
- underlined terms denote the dissipative, viscous terms (or turbulent counterparts).
- We consider a cylindrical domain D with radius $= r\sqrt{x^2+y^2} \in [0,R]$ and, on average, a vertical coordinate $z \in [0, H_T]$ for fixed R and H_T.

4 3D Baroclinic Quasigeostrophy

- Ignoring the underlined dissipative terms in (2), stratified quasigeostrophy arrives when hydrostatic balance is assumed (equating the twice underlined terms), and the buoyancy equation is used to eliminate $w_0 \propto b_1$ in the vertical vorticity equation:

$$
\begin{aligned}
\partial_t q + J(\psi, q) &= 0 && \text{(3a)}\\
q &\equiv \nabla^2\psi + \partial_z\Big(\frac{1}{\partial_z(\bar b_0 - \bar\rho/\hat\Gamma)}\partial_z\psi\Big) && \text{(3b)}\\
\partial_t\partial_z\psi + J(\psi,\partial_z\psi) &= 0 \quad \text{at} \quad z = 0, H_T && \text{(3c)}
\end{aligned}
$$

- with quasigeostrophic potential vorticity q.

5 Linear Dispersion Relations

Using $\propto e^{i(kx+ly+mz-\omega t)}$ with $\kappa^2 = k^2 + l^2$, for $\bar b_0$ constant, dispersion relations for the 3D parent Boussinesq, the reduced rotation constrained model, and the quasi-geostrophic equations are:

- Boussinesq: $\omega^2 = \frac{m^2/Ro^2 + \kappa^2(-\bar\rho')/Fr^2}{(\kappa^2+m^2)}$,
- rotation constrained: $\omega^2 = m^2/(\kappa^2 Ro^2) + (-\bar\rho'/Fr^2)$, arising when $m \ll k$, so for anisotropic convection:

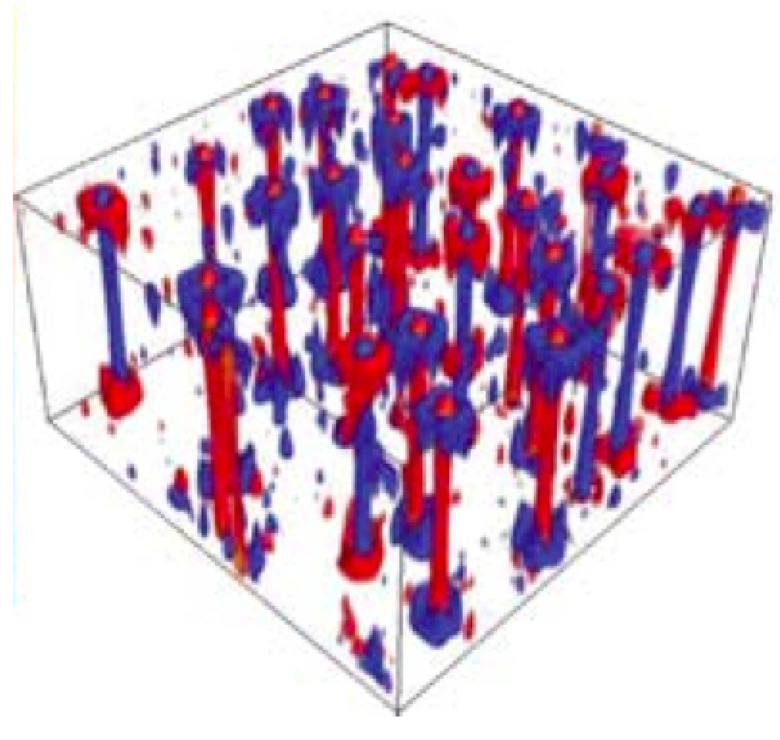

Fig. 1. *From Sprague et al. [5]: temperature/buoyancy anomaly. Their $Pr = 7, \tilde{Ra} = 40$.*

- quasigeostrophy: $\omega = 0$.

6 Hamiltonian Formulation

- In the inviscid case, the Hamiltonian/energy of (2) is:

$$
\mathcal{H} = \frac{1}{2}\int_D |\nabla_H\psi|^2 + w_0^2 + \frac{\hat\Gamma}{\partial_z(\bar b_0 - \rho/\hat\Gamma)} b_1^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad (4)
$$

upon using the boundary conditions $\psi = 0$ at $r = R$, $w_0 = 0$ at $z = 0, H_T$.

- Variations of the Hamiltonian are:

$$
\begin{aligned}
\delta\mathcal{H} &= \frac{1}{2}\int_D -\psi\delta\zeta + w_0\delta w_0 + \frac{\hat\Gamma}{\partial_z(\bar b_0 - \bar\rho/\hat\Gamma)} b_1\delta b_1 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z && \text{(5a)}\\
&= \int_D \frac{\delta\mathcal{H}}{\delta\zeta}\delta\zeta + \frac{\delta\mathcal{H}}{\delta w_0}\delta w_0 + \frac{\delta\mathcal{H}}{\delta b_1}\delta b_1 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z && \text{(5b)}
\end{aligned}
$$

using the restriction on the functional derivatives $\delta\mathcal{H}/\delta\zeta = 0$ at $r = R$ and $\delta\mathcal{H}/\delta w_0 = 0$ at $z = 0, H_T$.

- A co-symplectic formulation follows from (2) and (5a):

$$
\begin{aligned}
\partial_t\zeta &= J\Big(\frac{\delta\mathcal{H}}{\delta\zeta},\zeta\Big) + J\Big(\frac{\delta\mathcal{H}}{\delta w_0}, w_0\Big) + J\Big(\frac{\delta\mathcal{H}}{\delta b_1}, b_1\Big) + \partial_z\frac{\delta\mathcal{H}}{\delta w_0} && \text{(6a)}\\
\partial_t w_0 &= J\Big(\frac{\delta\mathcal{H}}{\delta\zeta}, w_0\Big) + \partial_z\frac{\delta\mathcal{H}}{\delta\zeta} + \partial_z(\bar b_0 - \rho/\hat\Gamma)\frac{\delta\mathcal{H}}{\delta b_1} && \text{(6b)}\\
\partial_t b_1 &= J\Big(\frac{\delta\mathcal{H}}{\delta\zeta}, b_1\Big) - \partial_z(\bar b_0 - \bar\rho/\hat\Gamma)\frac{\delta\mathcal{H}}{\delta w_0} && \text{(6c)}\\
\zeta &= \nabla^2\psi. && \text{(6d)}
\end{aligned}
$$

- Underlined terms are null. Potential vorticity (Julien et al. 2006) is conserved.

7 Numerical Weak Formulation

- Consequently, a weak formulation and candidate Hamiltonian formulation reads

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} &= \{\mathcal{F},\mathcal{H}\}\\
&\equiv \int_D \zeta J\Big(\frac{\delta\mathcal{F}}{\delta\zeta},\frac{\delta\mathcal{H}}{\delta\zeta}\Big) + w_0\Big(J\Big(\frac{\delta\mathcal{F}}{\delta w_0},\frac{\delta\mathcal{H}}{\delta\zeta}\Big) + J\Big(\frac{\delta\mathcal{F}}{\delta\zeta},\frac{\delta\mathcal{H}}{\delta w_0}\Big)\Big)\\
&\quad + b_1\Big(J\Big(\frac{\delta\mathcal{F}}{\delta b_1},\frac{\delta\mathcal{H}}{\delta\zeta}\Big) + J\Big(\frac{\delta\mathcal{F}}{\delta\zeta},\frac{\delta\mathcal{H}}{\delta b_1}\Big)\Big)\\
&\quad + \frac{\delta\mathcal{F}}{\delta\zeta}\partial_z\frac{\delta\mathcal{H}}{\delta w_0} - \frac{\delta\mathcal{H}}{\delta\zeta}\partial_z\frac{\delta\mathcal{F}}{\delta w_0}\\
&\quad + \partial_z(\bar b_0 - \bar\rho/\hat\Gamma)\Big(\frac{\delta\mathcal{F}}{\delta w_0}\frac{\delta\mathcal{H}}{\delta b_1} - \frac{\delta\mathcal{H}}{\delta w_0}\frac{\delta\mathcal{F}}{\delta b_1}\Big)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. && \text{(7)}
\end{aligned}
$$

- By using (5b) and the restriction on the functional derivatives $\delta\mathcal{H}/\delta\zeta = 0$ at $r = R$ and $\delta\mathcal{H}/\delta w_0 = 0$ at $z = 0, H_T$ (for the inviscid case), the Hamiltonian formulation (7) yields the (inviscid) equations of motion (2), essentially by reversing its construction.
- Again $\zeta = \nabla^2\psi$ must be defined and used in separation.
- Clearly (7) is skew-symmetric and Jacobi's identity $\{\mathcal{F},\{\mathcal{G},\mathcal{H}\}\} + \{\mathcal{H},\{\mathcal{F},\mathcal{G}\}\} + \{\mathcal{G},\{\mathcal{H},\mathcal{F}\}\} = 0$ *verified* (doubly-periodic horizontal domain).
- If we take variations of $\mathcal{F}$ as arbitrary test functions, then (7) serves as (finite element) weak formulation.

8 Phase Changes: Iodine Cycle

- Consider a container with dry air at room temperature and a small mass fraction of solid iodine particles on the bottom.
- Heat and keep the bottom above the iodine sublimation temperature $T_s = 386$K.
- Keep the top below $T < Ts$ with a teflon surface repelling iodine solidification.
- Rotating Rayleigh-Bénard convection set-up.
- Total dimensional density is related to temperature as follows: $\rho = \rho_0(1 - \alpha_T T)$.
- A bulk two-state moisture model is adopted with iodine vapor q_v and iodine snow/precipitate q_s, cf. similar approach in Zerroukat & Allen.

9 Future Work: Conceptual Laboratory Experiment & DA

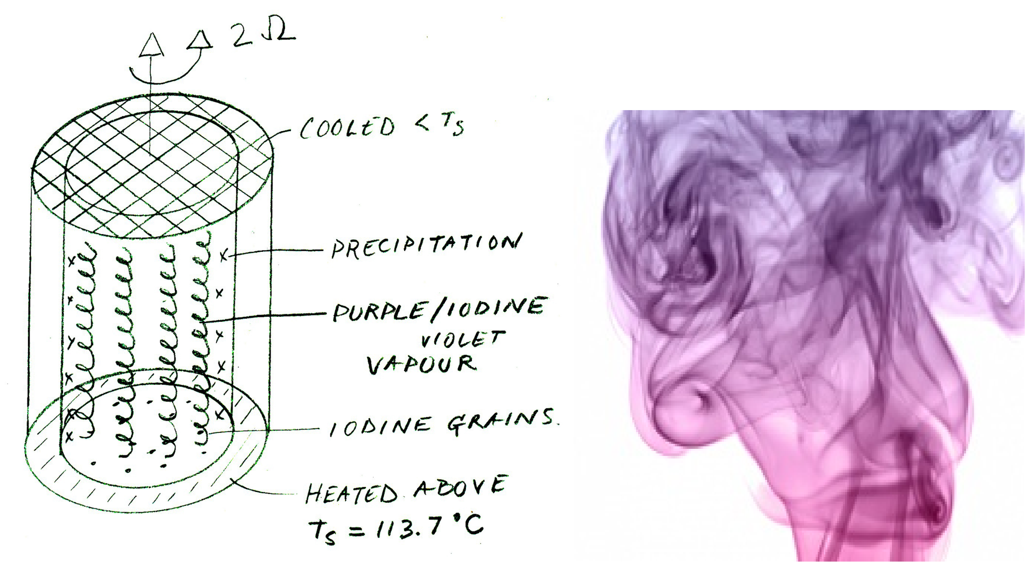

Fig. 2. *Left: Sketch of experimental set-up for rotating Rayleigh-Bénard convection with iodine phase changes. Right: sample of iodine vapor.*

References

[1] http://www1.maths.leeds.ac.uk/~obokhove
http://www1.maths.leeds.ac.uk/~smt
http://www1.maths.leeds.ac.uk/~mmtk

[2] Julien et al. 2006: Generalized quasi-geostrophy for anisotropic rotationally constrained flows. JFM 555.

[3] Kent, Tobias, Bokhove 2015: A modified 1.5D shallow water model for investigating convective-scale data assimilation. In prep. *MWR*.

[4] Lorenz, E. 2005: Designing chaotic models. *JAS* 62.

[5] Sprague, M., Julien, K., Werne, J. 2006: Numerical simulation of an asymptotically reduced system for rotationally constrained convection. JFM 551.

[6] Zerroukat, M. Allen, T. 2015: A moist Boussinesq shallow water equations set for testing atmospheric models. JCP 290.